Model for Long-Wave Radiation

CLIM 710
Introduction to the Physical Climate System

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Here a simple analytic model for *long-wave radiation* is described. We start from the description of the change in spectral radiance L_ν along a distance ds along a particular direction, as described in Part II of the lecture on Radiation and Climate. From equation (9) in that chapter:

$$-\frac{1}{k_\nu}\frac{1}{\rho_a(s)}\frac{dL_v}{ds} = L_\nu - J_\nu = L_\nu - B_\nu \tag{1}$$

where k_ν is the extinction coefficient, ρ_ν is the density of absorbers relevant for the particular frequency, and the source term J_ν is replaced by the Blackbody emission radiance B_ν as described in Part II.

Radiative Depth

We want to describe radiation upwelling from below a level plane to above it, or radiation downwelling from above a level plane to below it.

Consider a beam of radiation upwelling from the lower troposphere to the upper troposphere. Let $\mu = \cos(\theta)$, where θ is the angle between the (upward) direction of the beam and the vertical. Then $\theta = 0$ ($\theta = \pi$) corresponds to upward (downward) directed radiation. Then:

$$dz = \cos(\theta)\, ds = \mu\, ds \tag{2}$$

and equation 1 then becomes

$$-\frac{\mu}{k_\nu}\frac{1}{\rho_a(s)}\frac{dL_\nu}{dz} = \mu\frac{dL_\nu}{d\tau_\nu} = L_\nu - B_\nu \tag{3}$$

where we have introduced the *radiative depth* defined as:

$$\tau_\nu(z) = \int_z^\infty k_\nu(z')\rho_a(z')\, dz' \tag{4}$$

The radiative depth is analogous to optical depth, but defined in the vertical direction. Note that on the right-hand side of equation 3, L_ν is now a function of τ for any specified direction, and so is $B_\nu(T)$ once we know the vertical distribution of temperature T.

We introduce the approximation that the atmospheric properties k and ρ are *independent of the frequency* ν, thus assuming a *gray atmosphere*. Hence $\tau_\nu(z) = \tau(z)$. Multiplying equation 3 by μ:

$$\mu^2 \frac{dL_\nu}{d\tau} = \mu L_\nu - \mu B_\nu \tag{5}$$

Integration over angles

For upwelling radiation, we integrate equation 5 over all azimuthal angles ϕ ($0 \leq \phi \leq 2\pi$) and over values of μ so that $0 \leq \theta \leq \pi/2$, or $0 \leq \mu \leq 1$. [1]

$$\frac{d}{d\tau} \int d\phi \int_0^1 d\mu\, \mu^2 L_\nu = \int d\phi \int_0^1 d\mu\ \mu L_\nu - \int d\phi \int_0^1 d\mu\ \mu B_\nu \quad (6)$$

For downwelling radiation, the procedure is the same, but now $-1 \leq \mu \leq 0$:

$$\frac{d}{d\tau} \int d\phi \int_{-1}^0 d\mu\, \mu^2 L_\nu = \int d\phi \int_{-1}^0 d\mu\ \mu L_\nu - \int d\phi \int_{-1}^0 d\mu\ \mu B_\nu \quad (7)$$

[1] $d\phi d\mu = d\phi \sin(\theta) d\theta$ gives the correct integration over solid angles in spherical coordinates.

The extra factor of $\mu = \cos(\theta)$ in each of the two integrals on the right-hand side of both equations 6 and 7 is necessary to convert the spectral radiance to spectral irradiance, as described in a previous lecture. Thus

$$\int d\phi \int_0^1 d\mu \ \mu L_\nu = F_\nu^\uparrow(\tau) \quad (8)$$

$$\int d\phi \int_{-1}^0 d\mu \ \mu L_\nu = F_\nu^\downarrow(\tau) \quad (9)$$

give the upwelling and downwelling spectral irradiance, respectively.

Further, on the right hand sides of equations 6 and 7, B_ν is independent of angular direction, so the last term in each equation becomes simply πB_ν.

We now assume that the upwelling spectral radiance L_ν is *isotropic*, or independent of direction. Then using equations 8 and 9:

$$F_\nu^\uparrow(\tau) = L_\nu(\tau,1)\int d\phi \int_0^1 d\mu \; \mu = \pi L_\nu(\tau,1) \qquad (10)$$

$$F_\nu^\downarrow(\tau) = L_\nu(\tau,-1)\int d\phi \int_0^1 d\mu \; \mu = \pi L_\nu(\tau,-1) \qquad (11)$$

where $L_\nu(\tau,\pm 1)$ means the value of L_ν at $\mu = \pm 1$.

Isotropic Radiation - continued

Using the property of isotropy on the left-hand side of equations 6 and 7, and integrating over ϕ:

$$2\pi \int_0^1 d\mu\, \mu^2 L_\nu = 2\pi L_\nu(\tau, 1) \int_0^1 d\mu\, \mu^2 = \frac{2\pi}{3} L_\nu(\tau, 1) = \frac{2}{3} F_\nu^{\uparrow}(\tau) \tag{12}$$

$$2\pi \int_{-1}^0 d\mu \mu^2 L_\nu = 2\pi L_\nu(\tau, -1) \int_{-1}^0 d\mu \mu^2 = -\frac{2\pi}{3} L_\nu(\tau, -1) = -\frac{2}{3} F_\nu^{\downarrow}(\tau) \tag{13}$$

Isotropic Radiation - Integrate over ν

Using equations 8, 9, 12 and 13 in equations 6 and 7 gives:

$$\frac{2}{3}\frac{d}{d\tau}F_\nu(\tau)^\uparrow = F_\nu^\uparrow(\tau) - \pi B_\nu \quad (14)$$

$$-\frac{2}{3}\frac{d}{d\tau}F_\nu(\tau)^\downarrow = F_\nu^\downarrow(\tau) - \pi B_\nu \quad (15)$$

Integrating over all frequencies, we have the definitions:

$$F^\uparrow(\tau) = \int_0^\infty d\nu F_\nu^\uparrow(\tau)$$

$$F^\downarrow(\tau) = \int_0^\infty d\nu F_\nu^\downarrow(\tau)$$

and the straightforward result that:

$$\pi B(T) = \int_0^\infty d\nu \pi B_\nu = \sigma_B T^4 \quad (16)$$

Final Radiation Equations

Using the above definitions and results in equations 14 and 15:

$$\frac{2}{3}\frac{d}{d\tau}F^{\uparrow}(\tau) = F_{\nu}^{\uparrow}(\tau) - \pi B_{\nu} \quad (17)$$

$$-\frac{2}{3}\frac{d}{d\tau}F^{\downarrow}(\tau) = F_{\nu}^{\downarrow}(\tau) - \pi B_{\nu} \quad (18)$$

The *Boundary Conditions* for long-wave radiation for the top of the atmosphere ($\tau = 0$; $z = \infty$) and bottom ($\tau = \tau_s$; $z = 0$) are:

$$F^{\downarrow}(\tau = 0) = 0 \quad (19)$$

$$F^{\uparrow}(\tau = \tau_s) = \pi B(T_s) = \sigma_B T_s^4 \quad (20)$$

Radiative Equilibrium

The *net* irradiance in the upward direction (energy per unit time per unit area across a horizontal plane) is

$$F(\tau) = F^{\uparrow}(\tau) - F^{\downarrow}(\tau) \quad (21)$$

Multiplying F by a small volume $dz\, dA$ gives the net heating, which is equal to the product of the mass of the volume $\rho dz\, dA$ times the heating per unit mass Q:

$$(F^{\uparrow} - F^{\downarrow})\, dz\, dA = F\, dz\, dA = Q\rho\, dz\, dA \quad (22)$$

In radiative equilbrium, the net heaing Q vanishes in the interior:

$$F = (F^{\uparrow} - F^{\downarrow}) = C \quad (23)$$

where C is a constant.

Application to the Earth-Atmosphere System

In order for the whole earth-atmosphere system to be in radiative equilibrium, the outgoing long-wave radiation at the top of the atmosphere must balance the incoming solar irradiance (integrated over all frequencies) which we define as the constant F_0. Then we have another boundary condition at the top ($\tau = 0$):

$$(F^{\uparrow} - F^{\downarrow}) = F^{\uparrow} = F_0 \tag{24}$$

Thus the constant C appearing in equation 23 is just F_0.

Taking the sum and difference of equations 17 and 18 as (17 $+$ 18) and (17 - 18) we obtain:

Solution of Equations

$$\frac{2}{3}\frac{d}{d\tau}(F^{\uparrow} - F^{\downarrow}) = F^{\uparrow} + F^{\downarrow} - 2\pi B \quad (25)$$

$$\frac{2}{3}\frac{d}{d\tau}(F^{\uparrow} + F^{\downarrow}) = F^{\uparrow} - F^{\downarrow} = F_0 \quad (26)$$

where we have used equation 23 in the second equation. In equation 25 the term $(F^{\uparrow} - F^{\downarrow})$ is constant, so its τ derivative vanishes, and this equation becomes:

$$F^{\uparrow} + F^{\downarrow} = 2\pi B \quad (27)$$

Equation 26 is easily integrated in τ:

$$\frac{2}{3}(F^{\uparrow} + F^{\downarrow}) = F_0\tau + D \quad (28)$$

where D is a constant. Evaluating equation 28 at the top $\tau = 0$ and using boundary conditions 19 and 24 gives $\frac{2}{3}F_0 = D$.

Solution - continued

Using this value of D in 28 yields:

$$\frac{2}{3}(F^{\uparrow} + F^{\downarrow}) = F_0(\tau + \frac{2}{3}) \tag{29}$$

or

$$(F^{\uparrow} + F^{\downarrow}) = F_0(\frac{3}{2}\tau + 1) \tag{30}$$

Repeating equation 23 with $C = F_0$:

$$(F^{\uparrow} - F^{\downarrow}) = F_0 \tag{31}$$

Taking (30 $+$ 31) and (30 - 31) one gets the solutions to $F^{\uparrow}$ and $F^{\downarrow}$ in terms of τ:

Solution - continued

$$F^{\uparrow} = \frac{F_0}{2}(\frac{3}{2}\tau + 2) \quad (32)$$

$$F^{\downarrow} = \frac{F_0}{2}\frac{3}{2}\tau \quad (33)$$

which gives the solution of the irradiances in terms of τ.

Using equation 30 in $(F^{\uparrow} + F^{\downarrow}) = 2\pi B$ from equation 27 one easily obtains:

$$\pi B = \sigma_B T^4 = \frac{1}{2}F_0(\frac{3}{2}\tau + 1) \quad (34)$$

giving the temperature T as a function of radiative depth τ.

Radiative Balance at the Ground

We evaluate the solution for $F^{\uparrow}$ at the ground ($z = 0$ or $\tau = \tau_s$) from equation 32:

$$F^{\uparrow}(z = 0) = \frac{F_0}{2}(\frac{3}{2}\tau_s + 2) \quad (35)$$

We define an equivalent black body temperature of the surface T_e as:

$$\sigma_B T_e^4 = F^{\uparrow}(z = 0) = F_0(\frac{3}{4}\tau_s + 1) \quad (36)$$

The factor $(\frac{3}{4}\tau_s)$ quantifies the greenhouse effect; for without an atmosphere, the earth's equivalent black body temperature would simply be given by $\sigma_B T_e^4 = F_0$

Air Temperature at the Ground

We obtain the air temperature T_s at $\tau = \tau_s$ directly from equation 34:

$$\pi B = \sigma_B T_s^4 = \frac{1}{2} F_0 (\frac{3}{2} \tau_s + 1) \tag{37}$$

Comparing to equation 36, we have:

$$\sigma_B (T_e^4 - T_s^4) = \frac{1}{2} F_0 \tag{38}$$

In this model, T_e is the surface temperature. The discontinuity between the surface temperature and the colder air temperature at the surface indicates this solution is *not realizable*, since atmospheric convection would act to warm the air near the surface to above its radiative equilibrium value.

Figure 3.20 Results from a simple two-stream model. The sloping lines show the variations with scaled optical depth χ^* of the upward ($F^\uparrow$) and downward ($F^\downarrow$) spectrally integrated long-wave irradiances and of the spectrally integrated black-body irradiance πB.

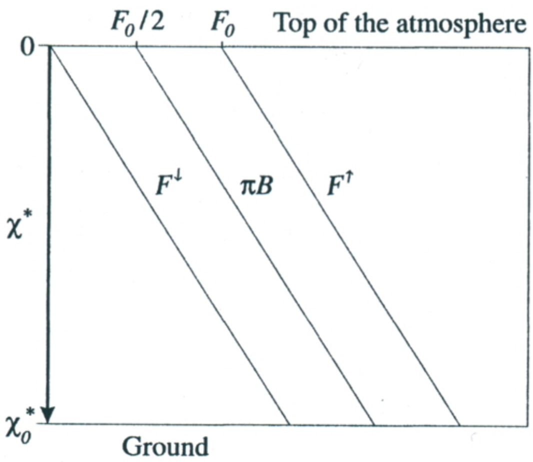

Figure 1: **Solution of Long Wave Model (from Andrews).**